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Iteration as an Avenue for Mathematical Exploration

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Iteration as an Avenue for Mathematical Exploration

by

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Report

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Abstract

Iteration as an Avenue for Mathematical Exploration

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This report explores several applications of iteration and the various connections that can be made to different areas of mathematics. The ties iteration has to the Wada Property, bifurcation diagram, root finding, and applications in geometry are all investigated. Finally, a rationale for incorporating iteration into secondary mathematics courses to support a more robust curriculum is discussed.

Table of Contents

List of Figures

Chapter 1: Introduction

Simply put, *iteration* is repeating a process over and over again, where the output of one step is used as the input for the next. One can iterate geometrically by performing an action, such as creating the Cantor set by removing the middle third of a line segment, as shown in Figure 1, or by repeatedly connecting the midpoints of a triangle to form Sierpinksi's triangle. Algebraic iteration of certain quadratic functions with real and complex parameters yields the bifurcation diagram, which has many interesting features as well as connections to the Mandelbrot fractal.

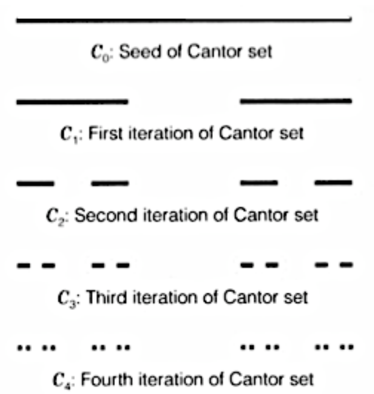

Figure 1. Four iterations of the Cantor set [**5**, p. 261].

Iteration begins with a function and an initial seed value, x, where $F^n(x) = F(F^{n-1}(x))$, and n indicates the number of iterations of the function. The resulting set of $F^n(x)$ is called the *orbit* of the seed value x. A question that follows is what happens to the orbit of x as the process of iteration continues. For example, the behavior of the function $F(x) = x^2 - 2$ under iteration is sensitive to the initial starting conditions [**3**, p. 33]. If the function is iterated with seed values of 0, 0.1, 0.01, and

0.001, the fate of the orbits are very different; the orbit of 0 converges to 2 after two iterations, but the orbits of the others do not converge. Because the orbit values must be rounded, different rounding will produce different orbits for the same seed value. Although there exists a specific rule, the orbit of *x* will vary depending on accuracy, making the system impossible to predict. [**3**, p. 33].

In order to understand the results of iteration, one must understand what happens to the orbit of a seed value. To do this, some basic information is necessary. A point *w* is a *fixed point* of a function *f* if $f(w) = w$. This point is *attracting* if $|f'(w)| < 1$, *super-attracting* if $f'(w) = 0$, *neutral* if $|f'(w)| = 1$, or *repelling* if $|f'(w)| > 1$. If $w_{k+1} = f(w_k)$ for $k = 1, 2, \ldots, n-1$, and $w_1 = f(w_n)$, then the orbit $\{w_1, w_2, \ldots, w_n\}$ is a *cycle of period n* for *f*. If $w_1 \neq f(w_j)$ for $j < n$, the cycle has *prime period n* [**1**, p. 106]. In 1975 Tien-Yien Li and James A. Yorke proved that if a continuous function on a closed interval has a point with period 3, then the function must also have a point with every other period [**2**, p. 162]. However, this is one special case of Sharkovski's theorem. In 1962, Alexandr Sharkovski created a unique ordering system of the natural numbers and proved that if a function has a period *k*, then it must have periods of all numbers that come after *k* in the ordering system [**2**, p. 161].

The basic concept of iteration is straightforward enough to be included in the typical high school curriculum. Iteration has ties to compositions of functions, finding roots, geometric constructions, and fractals, all of which are part of the secondary mathematics curriculum. By investigating fractals, students can develop algebraic functions to describe the area and perimeter of the Koch snowflake and Sierpinksi's triangle and the length of the Cantor set. Students can also investigate the connection between complex numbers and geometry by exploring the Mandelbrot set, and discuss limits by observing that the total length of the Cantor set approaches zero, but never reaches it [**5**, p. 265]. The orbit of a seed value can also be represented by *graphical iteration* as shown in Figure 2, which is a concept secondary students can grasp and explore. To *graphically iterate,* an initial value is chosen and a vertical line is drawn to the function. Then, from this point, a horizontal line is drawn to the line $y = x$ so that the output becomes the new input. Finally, a vertical line is drawn back to the function, and the whole process is repeated as many times as necessary. *Newton's method* can be explored geometrically and algebraically, and with the abundance of computer graphing software, students can apply *Newton's method* to find roots of functions they might not be able to solve for otherwise.

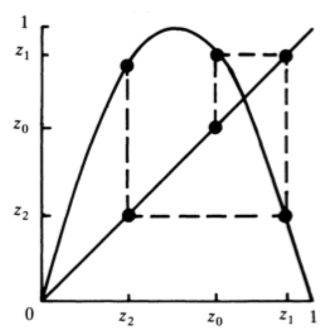

Figure 2. Graphical iteration [**9**, p. 9].

Exploring iteration leads to many surprising discoveries and allows one to make connections between many branches of mathematics; a feature that is sorely lacking in secondary education. Several applications and results of iteration will be discussed, illustrating that iteration is an accessible avenue through which different fields of mathematics can be explored.

Chapter 2: The Wada Property

One of the most common applications of iteration, *Newton's method*, can be used to find the roots of a function *f*. To solve for the zeros of *f*, an initial guess, x_0, is made and a sequence of tangent lines whose *x*-intercepts approach the zero is generated by the iterative process. Algebraically, the zeros of *f* can be approximated using the *Newton function,*

$$N_f(x) = x - \frac{f(x)}{f'(x)},$$

where $x_1 = N_f(x_0)$, $x_2 = N_f(x_1)$, … , $x_n = N_f(x_{n-1})$ until x_n is as a close of an approximation of the root as necessary. Using Newton's method to solve for the zeros of *f* is equivalent to using graphical iteration to find the fixed points of N_f [**4**, p. 194].

The collection of *x* values that iterate to a particular zero, z_*, is called the *basin of attraction* of z_*, denoted $B(z_*)$. The largest interval subset of $B(z_*)$ that contains z_* is the *immediate basin of attraction* of z_*, which is written as $B_0(z_*)$ [**4**, p. 195]. When applying Newton's method to a function, the basins of attraction are not known, but they can be found by reversing the graphical iteration process. The basin of attraction of z_* is the union of all intervals obtained by applying reverse graphical iteration to the immediate basin of z_*. In other words, $B(z_*) = \cup N_f^{-1}(B_0(z_*))$ [**4**, p. 198]. To find the basins of attraction for the zeros of *f*, reverse graphical iteration is applied to the graph of N_f. Surprisingly, the basins of attraction of the zeros of a function satisfy a unique property called the *Wada Property,* which will be explained below [**4**, p. 192.

To visualize the *Wada Property,* imagine trying to paint a picture with the colors red, yellow, and blue, and ensuring that between every two regions of a different color, there exists a region of the third color. Another example illustrating the *Wada Property* is called the *Lakes of Wada,* which was first reported by Yoneyama [**4**, p. 199]. In this illustration, there are three canals dug into an island. One canal comes from the ocean, represented by the large, dark rectangle in Figure 3, and the other two come from the lakes, each one square unit in area. The first canal is constructed from the ocean so that every point on the island is no more than $\frac{1}{2}$ unit from that canal. A second canal is dug from one of the lakes so that every point on the island is no more than $\frac{1}{4}$ unit away. Then, a third canal is dug so that every point on the island is no more than $\frac{1}{8}$ unit away. As this process continues from the outer edge of the rectangle again, the island is mostly replaced with canals, and between any two canals, there exists the third, therefore exhibiting the *Wada Property* [**4**, p. 199]. Formally, when three of more pairwise disjoint sets are arranged so the boundary point of two of the sets is a boundary point of all of them, the sets are said to satisfy the *Wada Property* [**4**, p. 199].

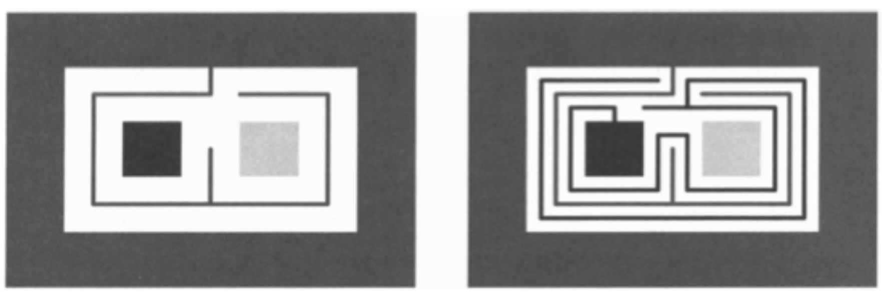

Figure 3. Canals exhibiting the Wada Property [**4**, p. 199].

To see how the basins of attraction of a function satisfy this property, two facts are needed. First, Frame and Neger show that the zeros of finite multiplicity of $f(x)$ are the same as the fixed points of N_f, and that these fixed points are attracting [**4**, p. 195]. Second, if a function has zeros $z_1 < z_2 < \ldots < z_k$ and critical values $c_1 < c_2 < \ldots < c_{k-1}$, then the immediate basins of attraction for z_1 and z_k are $B_0(z_1) = (-\infty, c_1)$ and $B_0(z_k) = (c_{k-1}, \infty)$. For all other zeros, z_i, the immediate basin of attraction is the interval (l_i, r_i) on the x-axis where l_i and r_i are the two cycle points for N_f; that is $l_i = N_f(r_i)$ and $r_i = N_f(l_i)$ [**4**, p. 198].

The following example illustrates how to find the basins of attraction for the zeros of a polynomial. The basin of attraction for the root $x = 0$ of the polynomial $f(x) = (x-2)(x-1)x(x+1)(x+2)$ can be found by examining the corresponding fixed point $x = 0$ of N_f, which is shown in Figure 4 [**4**, p. 196]. By the previous observation above, the immediate basin of attraction for this fixed point is the interval on the x-axis enclosed by the 2-cycle points, indicated by the center set of dashed lines surrounding $x = 0$ in Figure 4(a).

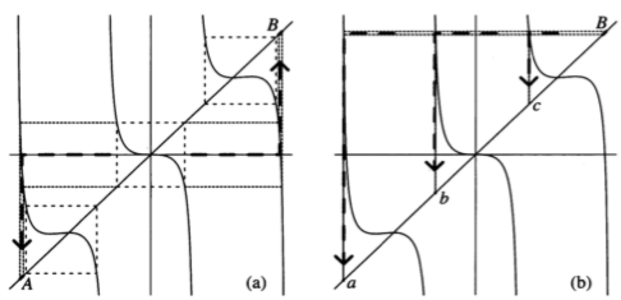

Figure 4. Graph of N_f and basins of attraction for $x = 0$ [**4**, p. 199].

To find additional intervals in the basin of attraction, reverse the graphical iteration process by extending the dashed lines horizontally to the graph and vertically to the diagonal to find two additional intervals, ***A*** and ***B***, as shown in Figure 4(a) [**4**, p. 198]. Applying reverse graphical iteration to either of these intervals yields three additional intervals, a, b, and c, and from each of these, three more intervals can be found. See Figure 4(b). This process can be repeated to find the basins of attraction for the other fixed points of N_f, whose immediate basins are indicated by the left and right dashed boxes in Figure 4(a) [**4**, p. 199]. Taken collectively, these basins have a complicated arrangement and the arrangement satisfies the Wada Property [**4**, p. 200].

To show the basins of attraction for certain polynomials satisfy the Wada Property, one must first show that some amount of space separates intervals of two basins. Then, it can be proven that between any two intervals, an interval of another basin exists, which establishes the property [**4**, p. 201].

To show two intervals are some distance apart, suppose (a_i, b_i) and (a_j, b_j) are intervals in the basins of attraction for the zeros z_i and z_j with $b_i \le a_j$ [**4**, p. 200]. This implies that each interval is a component of the immediate basin for z_i or z_j. Assume these roots are not the first or last roots. The alternate case where the roots are the first and last is treated similarly. Then, by the earlier observation, $B_0(z_i) = (l_i, r_i)$ and $B_0(z_j) = (l_j, r_j)$. Graphically iterating m times from a_i or b_i will yield l_i or r_i, and likewise, iterating n times from a_j or b_j yields l_j or r_j depending on the parity of m and n [**4**, p. 201]. Suppose $b_i = a_j$ and $n = m + k$. Then the following hold [**4**, p. 201],

$$N_f^n(a_j) = l_j \text{ or } r_j\text{, and}$$

$$N_f^n(b_i) = N_f^{m+k}(b_i) = N_f^k(\text{one of } l_i \text{ or } r_i) = \text{one of } l_i \text{ or } r_i\,.$$

This implies that one of l_i or r_i equals one of l_j or r_j, but this is impossible since immediate basins of z_i and z_j are separated by vertical asymptotes, which means the intervals (a_i, b_i) and (a_j, b_j) must have some distance between them [**4**, p. 201]. This leads to the following theorem [**4**, p. 201]:

Theorem 1. *If f is a polynomial with all real zeros and at least four different zeros, then the basins of attraction of the zeros of f have the Wada Property.*

To show these intervals exhibit the Wada Property, suppose (t,u), (r,s), and (v,w) are all intervals with (t,u) between the other two. Applying reverse graphical iteration to all three intervals yields several components of each, with a component of $N_f^{-1}(t,u)$ between a component of $N_f^{-1}(r,s)$ and $N_f^{-1}(v,w)$ due to the fact that N_f is monotonic outside 2-cycle intervals [**4**, p. 201]. If $k \neq i \neq j$, then between $B_0(z_i)$ and $B_0(z_j)$ there exists a component (t,u) of $N_f^{-1}(B_0(z_k))$. Applying this reverse graphical iteration to components of $N_f^{-1}(B_0(z_i))$ and $N_f^{-1}(B_0(z_j))$ will always yield a component of $N_f^{-q}(t,u)$ between them [**4**, p. 202].

Making the connection between the zeros of a function and the fixed points of the related Newton function gives interesting and surprising results when a simple process is applied backwards. Certain polynomials with real roots have basins of attraction that exhibit the Wada Property, but this property also appears when the basins of attraction of the complex roots for $f(z) = z^3 - 1$ are studied [**4**, p. 200]. The next application of iteration involving the bifurcation diagram illustrates how polynomial curves explain phenomena within the diagram.

Chapter 3: The Bifurcation Diagram

Upon examining the bifurcation diagram, one notices curves within the image, both bounding the image and embedded within, where iterates seem to be concentrated. This diagram can be created by iterating the function, $f_c(x) = x^2 + c$, with a seed value of $x_0 = 0$, and plotting the final iterates. However, by composing the function $f_c(x) = x^2 + c$ with itself, beginning from $x_0 = 0$, different polynomials can be formed. At each successive composition, a new polynomial is generated, and these polynomials are collectively known as the *Q-curves*. The *n*th polynomial is defined as $Q_n(c) = f_c^n(0)$ and after each composition the following polynomials are created: $Q_1(c) = c$, $Q_2(c) = c^2 + c, \ldots, Q_{n+1}(c) = (Q_n(c))^2 + c$ [**7**, p. 641]. For example, Q_4 represents the graph of all the fourth iterates of $x_0 = 0$ as a function of *c*. The first six *Q*-curves are graphed and shown in Figure 5.

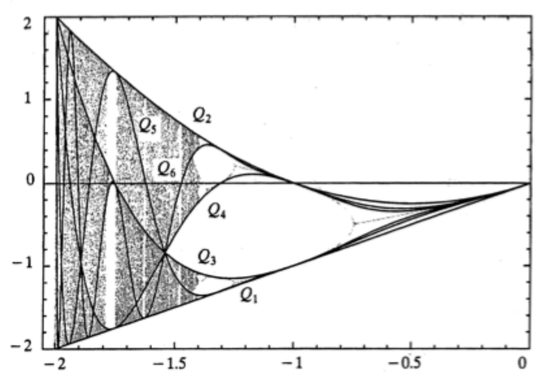

Figure 5. *Q*-curves [**7**, p. 641].

From Figure 5, it might appear that some *Q*-curves intersect the *x*-axis within certain *windows*, which are the vertical gaps of white space seen in the graph. Whenever a Q-curve intersects the x-axis, a window with period of the lowest numbered Q-curve that crosses there is found. These windows occur because each root of a Q-curve is a c value that has a super-attracting periodic cycle; points near c converge to the periodic cycle very quickly [**7**, p. 642]. Observations can also be made by examining the c values where two or more Q-curves intersect. There are only two types of intersections, those where one Q-curve is tangent to another, and those that are non-tangent. Tangency intersections occur at c values where a Q-curve crosses the x-axis, and therefore intersections of this type occur at windows. Non-tangent intersections occur at c values where f_c is chaotic [**7**, p. 645]. One of the main features of a chaotic system is "sensitivity on initial conditions", where small changes in the inputs yield very different outputs [**3**, p. 32]. The c values where the system is chaotic are called *Misiurewicz points*. At *Misiurewicz points*, the orbit of zero is *pre-periodic*; it will eventually fall into a periodic orbit that does not include zero [**7**, p. 644]. To show the intersection of curves occurs at either a root value or a Misiurewicz point, let c be the value where two Q-curves intersect. Because a Q-curve represents the iteration of zero, if c is a root, then the orbit of zero returns to itself, making it periodic. If c is not a root, the orbit doesn't return to zero. Since this c value represents an intersection with another Q curve, the orbit must become periodic at some point, which makes it a Misiurewicz point [**7**, p. 645]. These implications mean the bifurcation diagram is extraordinarily complex; there are a tremendous amount of non-visible windows representing periodic orbits as well as many areas of chaotic behavior.

The shape of the bifurcation diagram is also dictated by Q-curves, and these curves identify all areas where the diagram contains nested copies of itself [**7**, p. 647]. Each time a Q_n curve crosses the x-axis, self-similar copies of the diagram are bounded by Q_n and Q_{2n} and are made up of every nth iterate, while other iterates are contained between sets of different Q-curves. Each of these sets of curves where iterates are contained are called *envelopes* [**7**, p. 647]. For example, the whole diagram is bounded by Q_1 and Q_2 and all even iterates are bounded by Q_2 and Q_4 up until $-Q_2$ intersects with Q_4. This implies all odd iterates are bounded by Q_1 and Q_3 within this same range of c values [**7**, p. 647]. In the period three window, as shown in Figure 6, note Q_3 crosses the axis within this window and therefore another copy of the diagram is bounded between Q_3 and Q_6 as is every third iterate. Other iterates are confined to regions between Q_1 and Q_4 or Q_2 and Q_5, hence the visual separation of the period three window [**7**, p. 648]. In Figure 6, notice that the window ends where $-Q_3$ meets Q_6, which is a Misiurewicz point, s [**7**, p. 648]. At each intersection of a Q_n with the x-axis there are more and more self-similar copies of the diagram nested within the original.

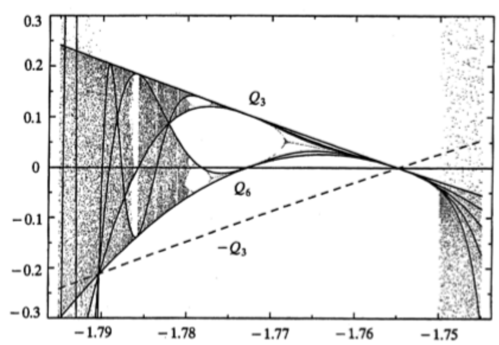

Figure 6. Envelopes in the period 3 window [**7**, p. 648].

To prove all iterates are contained within these Q-curve envelopes on $[s, r]$, where r is the root of the smallest Q_n that crosses there and s is a Misiurewicz point where $Q_{2n} = -Q_n$, one must "propagate the bounds" from one iterate to the next [**7**, p. 649]. To do this, the requirement that $|Q_{2n}| < |Q_n|$ is needed, which can be proved, but can also be inferred from the graphs [**7**, p. 649]. Then, if an iterate x, lies between Q_m and Q_{m+n}, the next iterate will be between Q_{m+1} and Q_{m+1+n}, and if x lies between Q_n and Q_{2n}, then the next iterate lies between Q_1 and Q_{1+n}. Finally, proving Q_m and Q_{m+n} are nonzero and agree in sign in the interval $[r,s]$ is achieved by contradiction [**7**, p. 651].

The bifurcation diagram displays the long-term behavior of iterates of a function, and many distinctive features such as why windows of period *n* appear, where the system will behave chaotically, and why there are copies of the diagram nested within the original are explained by *Q*-curves. Sharkovski's ordering and locations of *hyperbolic* components of the Mandelbrot set are also uncovered in the bifurcation diagram. A periodic point *x,* with a cycle of period *n,* is *hyperbolic* if the cycle is either attracting or repelling. In the Mandelbrot set, these hyperbolic components are the regions in the complex plane where there is an attracting periodic cycle, and these components appear as buds attached to a larger section of the graph [**7**, p. 642]. The next example illustrates what happens when an iterative process is applied to a well-known quadratic and asked to solve for the impossible.

Chapter 4: Iteration and Root Finding

Newton's method can be used to approximate the real roots of a function. Sometimes, the method leads to the roots, and other times it does not. Applying Newton's method to the function $f(x) = x^2 + 1$ is a fruitless endeavor; there are no real roots to be found. However, there are examples of cycling, chaos, and divergence to infinity.

To highlight these different scenarios, the Newton function is applied to $f(x) = x^2 + 1$, which yields the Babylonian iteration for approximating square roots [**9**, p. 3]. The double angle trigonometric identities for sine and cosine are manipulated to fit the Babylonian iteration, $x_{n+1} = \frac{1}{2}\left(x_n - \frac{1}{x_n}\right)$, and the result is the following equation

$$\cot 2\theta = \frac{1}{2}\left(\cot\theta - \frac{1}{\cot\theta}\right),$$

where $x_0 = \cot\theta$ [**9**, p. 5]. Experimenting with different values of θ produces different results with the iteration. For example, if $\theta = \frac{\pi}{4}$, then iteration yields $\theta = \frac{\pi}{2}$ and $\theta = \pi$ in the first and second steps respectively. At this point, $\cot\theta$ is undefined, which is analogous to arriving at the minimum of $f(x) = x^2 + 1$ by applying Newton's method beginning with $x_0 = 1$. Iterating with $\theta = \frac{\pi}{3}$ yields a cycle, while iterating with an irrational $\frac{\theta}{\pi}$ leads to chaotic behavior [**9**, p. 6]. Again, there are equivalent starting x values for f that mimic this behavior, namely $x_0 = \frac{1}{\sqrt{3}}$, which gives rise to a 2-cycle, and $x_0 = 1000$, which leads to chaotic behavior [**9**, p. 4].

A connection can also be made between f and the logistic map by writing an iteration rule that examines the y-values of Newton's method applied to $f(x) = x^2 + 1$. Substituting the Babylonian iteration into $y_{n+1} = x_{n+1}^2 + 1$ and simplifying yields

$$y_{n+1} = \frac{1}{4}\left(\frac{y_n^2}{y_n - 1}\right),$$

which can further be simplified by letting $z = \frac{1}{y}$ [**9**, p. 7]. The new equation becomes

$$z_{n+1} = 4z_n - 4z_n^2$$

where z always lies between 0 and 1, and the points $x = 0$ and $x = \infty$ correspond to $z = 1$ and $z = 0$ respectively [**9**, p. 8]. When $z_0 = \frac{1}{2}$, iteration yields $z_1 = 1$, and $z_2 = 0$. All subsequent z's are zero. This is analogous to $x = \infty$ in the original iteration. If $z_0 = \frac{3}{4}$, then $z_1 = \frac{3}{4}$, and is therefore a fixed point, implying the y-value is constant. Converting $z_0 = \frac{3}{4}$ to x gives $x = \frac{1}{\sqrt{3}}$, which is a point with period 2. The other fixed point of $F(z) = 4z - 4z^2$ is $z = 0$, and both fixed points are repelling since $\left|F'(0)\right|$ and $\left|F'\left(\frac{3}{4}\right)\right|$ are both greater than 1 [**9**, p. 8]. To illustrate the repelling nature of the fixed points, choose z_0 close to zero. Then the next value of z will be about four times as big and z_2 is roughly $16z_0$. The orbit is quickly moving away from z_0, but since $z = \frac{3}{4}$ is repelling and points close to $z = \frac{3}{4}$ move twice as far away, the orbit will never converge [**9**, p. 9].

The function $4z - 4z^2$ belongs to the family of functions

$$z_{n+1} = az_n - az_n^2, \qquad 0 \le a \le 4$$

where, after iteration, the function diverges to negative infinity when $a > 4$. The fixed points of the function are $z = 0$ and $z = \frac{a-1}{a}$, which leads to $F'(0) = a$ and $F'\left(\frac{a-1}{a}\right) = 2 - a$. For the quadratic to converge to $z = 0$, a must satisfy $0 \le a \le 1$, and for convergence to $z = \frac{a-1}{a}$, a must satisfy $1 \le a \le 3$ [**9**, p. 9]. When $a > 3$, the iterates will not converge, but cycling may occur. The windows of stability for cycling values of a become smaller and smaller, and in fact, they are decreasing by the *Figenbaum factor* of 4.6692… [**9**, p. 11]. Finally, at $a = 4$, which is the iteration derived from $x_{n+1} = \frac{1}{2}\left(x_n - \frac{1}{x_n}\right)$, the function has non attracting cycles and chaos [**9**, p. 11].

The roots of the function $f(x) = x^2 + 1$ are $x = \pm i$, which can never be found using Newton's method. Clearly the method will fail, and converting the function to $\cot 2^n \theta$ shows the different ways this happens. Iteration in this case has ties to trigonometric identities and surprisingly, the logistic map. The final example shows how iteration can be applied in a geometric setting to solve an age-old geometric construction problem.

Chapter 5: Iteration and a Geometric Construction

The process of iteration can be used in a variety of geometric settings from trisecting an angle to generating fractal images. It is well known that the trisection of an angle using only a compass and straightedge is impossible, but Martins and Rodrigues show how this can be done using fixed-point iteration [**6**, p. 205]. To begin this iterative process, start with an arbitrary T_0 on $\overrightarrow{AL}$ and let E_0 be the intersection of $\overline{BT_0}$ and c. Using AB as the radius and E_0 as the center, construct a circle, and let T_1 be the intersection of this circle with $\overrightarrow{AL}$. This process is considered the basic step [**6**, p. 206]. Continue this process to obtain a sequence of points $T_0, T_1, T_2, \ldots$ on $\overrightarrow{AD}$. See Figure 7.

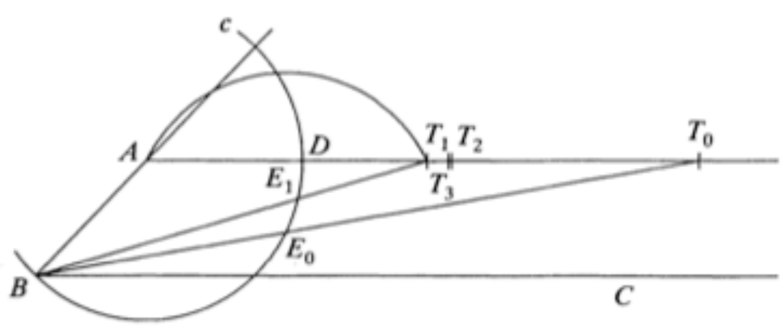

Figure 7. Iterative process [**6**, p. 206].

The basic step is defined as Φ, and applying Φ to T_0 yields the sequence $\{T_n\}$ and applying Φ to T yields T [**6**, p 206]. In order to show convergence of the sequence $\{T_n\}$, a coordinate system is applied to the construction and a function *f* is defined.

In the coordinate system, let *A* be located at the origin and the distance *AB* be equal to 1. See Figure 8. The function *f* is defined in the following way [**6**, p. 206]:

$$\text{if } P = (x, 0), \quad x > 0, \text{ then } \Phi(P) = \big(f(x), 0\big).$$

If $T_0 = (x_0, 0)$, then the sequence $\{x_n\}$ can be found by the iteration $x_{n+1} = f(x_n)$. Therefore, the convergence of $\{T_n\}$ is the same as the convergence of $\{x_n\}$ [**6**, p. 207]. To prove the convergence of $\{x_n\}$, a special case of the Contraction Mapping theorem is used [**6**, p. 207].

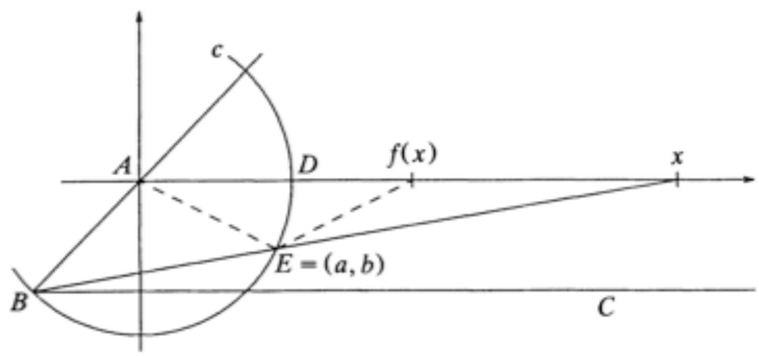

Figure 8. Coordinate system [**6**, p. 206].

This theorem relies on some interval *I*, and it can be seen from the construction that *f* maps $[1,2]$ onto itself [**6**, p. 207]. Let $[1,2]$ be the interval *I* and assume $x_0 \in I$ [**6**, p. 207]. Since *f* is continuous, all that remains to be shown is the existence of a $\gamma < 1$ such that $|f'(x)| \leq \gamma$ for all $x \in I$.

Because $f(x)$ is located at the intersection of a circle with $\overrightarrow{AL}$, where *E* is the center of the circle and *AB* is the radius, the distance from *E* to *A* is equivalent to the distance from *E* to $f(x)$. An isosceles triangle is formed, and the distance from *A* to $f(x)$ is $2a$, meaning $f(x) = 2a$. See Figure 8. Since $AB = 1$, and *c* is the unit circle, $B = (-\cos\alpha, -\sin\alpha)$. The following system for *a* and *b* is solved [**6**, p. 207]

$$a^2 + b^2 = 1$$
$$b = \frac{\sin\alpha}{x+\cos\alpha}(a - x)$$

where $\frac{\sin\alpha}{x+\cos\alpha}$ is the slope of the line through B and $(x,0)$, which leads to

$$f(x) = 2\cos\alpha + \frac{4x\sin^2\alpha}{1+2x\cos\alpha+x^2}. \quad (1)$$

To see if there exists a $\gamma < 1$ such that $|f'(x)| \le \gamma$ for all $x \in I$, the derivative and absolute value of (1) are taken [**6**, p. 207], which gives

$$|f'(x)| = \frac{4(x^2-1)\sin^2\alpha}{\left(1+2x\cos\alpha+x^2\right)^2}. \quad (2)$$

Because the right hand side of (2) is a non-decreasing function of α for $\alpha \in [0, \pi/2]$,

$$|f'(x)| \le \frac{4\left(x^2-1\right)}{\left(1+x^2\right)^2}$$

for $x \in [1,2]$ [**6**, p. 207]. The maximum of this expression occurs at $x = \sqrt{3}$, which can be found using graphing software, and the value is $\frac{1}{2}$. Therefore, $|f'(x)| \le \frac{1}{2}$ and so the sequence $\{x_n\}$ converges, which means that $\{T_n\}$ also converges. The iterative construction process will lead to a point T, such that $\overline{BT}$ trisects $\angle ABC$.

In this example, iteration was used in a geometric setting to accomplish something that cannot be done in a straightforward fashion. The construction is simple enough to be a suitable entry point for the concept of iteration in a secondary classroom since no symbolic manipulation is required and the construction can also be used to extend the topic of angle bisection. Once students grasp the concept of iteration, symbolic notation can be introduced and explored algebraically.

Chapter 6: Conclusion

Iteration is a fascinating topic through which to explore other areas of mathematics. Graphical iteration can be applied both forward to find the fixed points of a function and backwards to find basins of attraction. The Wada Property can also be extended to the complex plane when the basins of complex roots are studied. Plotting the long term iterates of quadratic functions gives rise to the bifurcation diagram whose secrets are unlocked by studying the curves that lie within, a surprising discovery in and of itself. Ties to fractal geometry and Sharkovski's ordering are a few of the other interesting features of the diagram. The more practical side of iteration, such as the ability to find roots via Newton's method, can be also be studied and different scenarios arise when there are no roots to be found. Applying an iterative process to a geometric construction provides a way to trisect an angle; a result that cannot be obtained outright. Surely coincidence doesn't explain the connections that can be found within these examples, such as the appearance of polynomials within the bifurcation diagram and the appearance of the logistic map while exploring $f(x) = x^2 + 1$, which makes the topic of iteration even more interesting and intriguing.

There is room for introduction of iteration at any level within the secondary curriculum. The mathematics curriculum in schools today does not highlight the usefulness or connectedness of the subject, and students leave the system thinking mathematics consists of applying archaic rules to manipulate numbers and variables. Students need to experience mathematics in ways other than applying the same algorithms to the same routine problems [**5**, p. 256]. The implications of iteration present myriad possibilities, many of which can be explored with ease using graphing or computer software. Algebra students can understand the basic graphical representation of Newton's method and geometry students can trisect an angle or explore area and perimeter of fractals. Students with more experience can begin to explore the finer details of iteration. The National Council of Teachers of Mathematics states that algebra students should be able to use iterative forms to represent relationships arising from various contexts [**8**]. Integrating iteration topics throughout the secondary curriculum would be a way to expose students to the world of mathematics, the one that exists outside the traditional setting, and is sure to spark interest and fascination among students of all levels and backgrounds.

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